Int. J. Solids Structures, 1973, Vol. 9, pp. 1053 to 1074. Pergamon Press. Printed in Great Britain

ON A FORMULATION OF THE BENDING OF ELASTIC PLATES

R. C. KOELLER

University of Colorado, Boulder, Colorado 80302

Abstract—A theory of the bending of linear elastic plates is developed in terms of an arbitrary set of linearly independent functions which forms a basis for the thickness parameter. A canonical system of equations is obtained in terms of stress matrix functions from which all kinematic variables, stress resultant and matrix stress resultants may be obtained. This canonical system of equations uses approximate transverse normal mode shapes which are obtained by solving an eigenvalue problem for the integrated expansion functions.

1. INTRODUCTION

THERE are several plate theories given in the literature which are formally equivalent to the linear theory of elasticity. One approach to an exact theory of plates is that given by Luré [1, 2] in which he develops a solution for the displacement field for the linear theory of elasticity in terms of the two-dimensional gradient operator and the thickness coordinate. Aksentian and Vorovich [3, 4] and Aksentian [5] apply asymptotic methods to this theory in order to solve the stress concentration problem of the circular hole in an infinite plate. Other procedures to obtain a general solution to the equations of linear elasticity within the spirit of plate theories have been through generalizations of the methods of series expansion of Cauchy [6] and Poisson [7] and the method of hypothesis of Kirchhoff [8]. Goodier [9] used an expansion in powers of the thickness of the plate in order to obtain a general solution in terms of a series of biharmonic functions in the case when the plate is subjected to edge tractions. He also obtains the Poisson–Kirchhoff theory when an approximation for the order of the thickness of the plate is carried out. An exact formulation for plates and cylinder subjected to edge loads was presented by Green [10] in terms of Fourier series, as well as power series, in terms of the thickness coordinate. Alblas [11], utilizing the work of Green [10], solved for the stress distribution in a thick plate containing a smooth circular cavity.

A different approach to the exact theory of plates has been developed by Tiffen and Lowe [12], where they define moments and higher order moments of stress and displacement with respect to the thickness coordinate. Several methods of approximation are then suggested in order to reduce the system of equations to a more tractable form. Mindlin and Medick [13] use a Legendre polynomial expansion in the thickness coordinate, followed by an integration across the thickness in the variational equation of motion, in order to reduce the equations of elasticity into an infinite series of two-dimensional equations. In a similar manner, Mindlin [14] develops a theory of vibration of crystal plates using a power series expansion and truncation of this theory, as well as [13], is performed by introducing correction factors.

As opposed to the theories which are developed from the three-dimensional theory of elasticity, the direct theory may be used. In the direct approach the plate or shell is regarded as a surface and one postulates a form of the balance laws. For the kinematics of the direct theory the surface, called a Cosserat surface, is regarded as a set of points and to each point a deformable vector, called a director, is assigned. For an explanation of the kinematics of oriented bodies, the development of the theory of shells and plates, as well as the history of the contributors, see the excellent monograph by Naghdi [15]. Green *et al.* [16] developed a general theory of a Cosserat surface utilizing fully consistent dynamical and thermodynamic principles of continuum mechanics. This theory was then specialized by Green and Naghdi [17] to the theory of an elastic Cosserat plate. A general nonlinear thermodynamic theory of shells using an infinite number of directors was developed by Green *et al.* [18] and Green and Naghdi [19]. They expanded the position vector of the deformed configuration in terms of a power series in the thickness coordinate and introduced various weighted averages in order to reduce the energy equation and the entropy production inequality in terms of surface quantities. The field equations for the surface were then obtained with the aid of invariance conditions under superposed rigid body motion.

In the present paper a generalization is made concerning the series expansion, insofar as the expansion is made in terms of a set of arbitrary functions. This generalization allows the set of functions to be selected by considerations which occur later in the development. The representations of [18, 19] are readily deduced except for the entropy inequality, and since the linear isothermal theory of the bending of elastic plates is being developed it is more expedient to develop the theory in terms of the linear theory of elasticity. In Section 2 the kinematics of the bending theory of plates are developed in terms of matrix notation, the equations of motion are given, and the constitutive equations for isotropic materials which are homogeneous with respect to the thickness coordinate are recorded. These results are the same as those given in [20], with the exception that the coefficients are known functions of the expansion functions. The Stokes–Helmholtz decomposition theorem is then applied in Section 3 in order to obtain a system of equations governing three stress matrix functions and one scalar function from which all the stress resultants and kinematical variables are obtained. This system of matrix equations is then canonicalized in Section 4. The resulting system of equations may be characterized as follows. One of the matrix equations is uncoupled from the other three equations and this matrix equation is diagonalized. The other two matrix equations are diagonalized but are coupled with the last scalar equation. In addition these equations are shown to be invariant with respect to the assumed form of the three-dimensional displacement field. The requirement that these equations be invariant under the assumed form of the displacement field is used to select the set of functions used in the series expansion for the three-dimensional displacement field. Several examples of these functions are then given. In Section 5 several restrictions are introduced in order to allow some comparisons with the work of Luré [1, 2], and in Section 6 an example is given so that a comparison with the theory of elasticity may be made.

Some remarks about the general scheme of notation should be given. In general the notation whenever convenient is that given in [15, 20]; however, there are some changes in notation that have been made. In this paper italic and greek letters are scalar, bold-face italic and greek letters are $P \times 1$ matrices and bold-face German letters are $P \times P$ matrices. Tensor indices having the range two are denoted by minuscule greek letters, the elements of matrices are denoted by majuscule italic letters and enumerative indices are denoted by minuscule German letters. The use of matrix notation is only used as a means for condensing the equations so that a bold-face letter with a tensor index should be interpreted as a set of P tensor equations.

2. BASIC EQUATIONS OF THE BENDING THEORY

Points in the plate are denoted by (X_1, X_2, X_3) where X_α, $\alpha = 1, 2$ are the plate coordinates and X_3 is the thickness coordinate. The thickness of the plate is denoted by h. Attention will be restricted to the bending of an elastic medium bounded by two parallel planes defined by $X_3 = \pm h/2$ where, without loss of generality, the (X_1, X_2) plane coincides with the middle plane of the plate. In addition, the elastic medium is bounded by a closed cylindrical surface. Nondimensional parameters are introduced through a characteristic parameter l in the plane (X_1, X_2). These nondimensional parameters are defined by

$$x_\alpha = X_\alpha/l, \qquad x_3 = 2X_3/h, \qquad H = h/2l, \tag{2.1}$$

so that H is a nondimensional half thickness parameter.

Following the customary and simplifying analysis of an elastic homogeneous isotropic medium bounded by two parallel planes, the boundary conditions on the planes $x_3 = \pm 1$ are considered to be prescribed stresses, and the decomposition into the symmetrical and asymmetrical loading of the plate is made. The asymmetrical loading problem results in the bending (flexure) problem in which the three-dimensional displacements u_1^* and u_2^* are odd functions and u_3^* is an even function of x_3. The symmetrical loading results in the extensional (stretching or compression) problem in which the three-dimensional displacements u_1^* and u_2^* are even functions and u_3^* is an odd function of x_3.

Since the bending theory is being developed and the elastic material is assumed to be homogeneous and isotropic, a solution to the linear theory of elasticity is sought in which the shear stress tensor components $t_{\alpha 3}^*$ and the transverse displacement u_3^* are even functions of x_3, whereas the stress tensor components in the plane of the plate $t_{\alpha\beta}^*$, the normal stress t_{33}^* and the displacement components in the plane of the plate u_α^* are odd functions in x_3. In the above and what follows, all quantities which depend on the three-dimensional space variables will be denoted by an asterisk, and all greek indices have the values of 1 or 2. Boundary conditions on the surfaces $x_3 = \pm 1$ are prescribed stresses which are such that they cause only bending; however, since the theory is linear, superposition may be used for the extensional part. Thus, the boundary conditions for the bending theory are taken as

$$\begin{aligned} t_\alpha^* &= \tfrac{1}{2}p_\alpha, \qquad & t_3^* &= \tfrac{1}{2}p_3 \quad \text{for } x_3 = 1, \\ t_\alpha^* &= \tfrac{1}{2}p_\alpha, \qquad & t_3^* &= \tfrac{1}{2}p_3 \quad \text{for } x_3 = -1, \end{aligned} \tag{2.2}$$

where p_α, p_3 are the prescribed surface tractions on the planes $x_3 = \pm 1$ and t_α^*, t_3^* are the components of the stress vector.

Let $\mathfrak{F}(P)$ denote the set of all functions defined in a closed interval $\mathfrak{L} = \{x_3| -1 \leq x_3 \leq 1\}$ which may be expanded in a power series of order $2P$ in x_3, i.e. $\mathfrak{F}(P) = \{f(x_3)| f(x_3) = \alpha_0 + \sum_{N=1}^{2P} \alpha_N x_3^N\}$, where P is an arbitrary integer. The set $\{1, x_3, x_3^2, x_3^3, \ldots x_3^{2P}\}$ forms a basis of the space $\mathfrak{F}(P)$. Now let $\Upsilon = \{1, \eta_1, \eta_2, \ldots \eta_P, \zeta_1, \zeta_2, \ldots \zeta_P\}$ be another linear independent basis of the space $\mathfrak{F}(P)$, where $\eta_N(x_3)$, $N = 1, 2, \ldots P$ are even functions and $\zeta_N(x_3)$, $N = 1, 2, \ldots P$ are odd functions of the nondimensional thickness coordinate $x_3 = 2X_3/h$. In what follows the set of functions $\{\eta_1, \eta_2, \ldots \eta_P\}$ and $\{\zeta_1, \zeta_2, \ldots \zeta_P\}$ will simply be denoted by $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ respectively. Introduction of $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ is made so that they may be selected in the course of the analysis in such a way as to simplify the resulting analysis. One important result is that there is a unique choice of these functions for a given value of P so as to make the resulting equations which govern the deflection of the plate to be invariant with respect to a change of bases. It is convenient to express x_3 in terms of the set $\boldsymbol{\zeta}$ and for $P \geq 1$ the identity function has a unique representation in terms of $\boldsymbol{\zeta}$ which is written as

$$x_3 = \sum_{N=1}^{P} \sigma_N \zeta_N(x_3) = \boldsymbol{\sigma}^T \boldsymbol{\zeta}, \tag{2.3}$$

where $\boldsymbol{\sigma}$ denotes the set of constants $\{\sigma_1, \sigma_2, \ldots \sigma_P\}$ and the above superscript T indicates the transpose.

It is now assumed that the three-dimensional displacement field for the bending of an elastic medium may be expressed in the form

$$\begin{aligned} u_\alpha^* &= \boldsymbol{\zeta}^T \boldsymbol{\gamma}_\alpha = \boldsymbol{\zeta}^T(\boldsymbol{\delta}_\alpha - H u_{3,\alpha} \boldsymbol{\sigma}), \\ u_3^* &= u_3 + \boldsymbol{\eta}^T \boldsymbol{\delta}_3, \end{aligned} \tag{2.4}$$

where $\boldsymbol{\gamma}_\alpha, \boldsymbol{\delta}_\alpha, \boldsymbol{\delta}_3$ are $P \times 1$ matrix functions and u_3 is a scalar function of the nondimensional plate coordinates and time. In addition partial differentiation with respect to the nondimensional plate coordinates x_α is denoted by a comma. It should be noted that the form (2.4) need not be a solution to all problems in the linear theory of elasticity restricted to bending problems. However, as shown for the static case the results are identical to that obtained by Luré [2] so that the above assumption appears to be justifiable.

The three-dimensional strain measures, $\gamma_{\alpha\beta}^*, \gamma_{\alpha 3}^*$ and γ_{33}^* are obtained from the assumed form of the displacement field (2.4) and are given by

$$\begin{aligned} l\gamma_{\alpha\beta}^* &= \tfrac{1}{2}(u_{\alpha,\beta}^* + u_{\beta,\alpha}^*) = \boldsymbol{\zeta}^T \boldsymbol{\kappa}_{(\alpha\beta)}, \\ 2l\gamma_{\alpha 3}^* &= \frac{1}{H}\frac{\partial u_\alpha^*}{\partial x_3} + u_{3,\alpha}^* = \frac{1}{H}(\boldsymbol{\zeta}')^T \boldsymbol{\delta}_\alpha + \boldsymbol{\eta}^T \boldsymbol{\kappa}_{3\alpha}, \\ l\gamma_{33}^* &= \frac{1}{H}\frac{\partial u_3^*}{\partial x_3} = \frac{1}{H}(\boldsymbol{\eta}')^T \boldsymbol{\delta}_3. \end{aligned} \tag{2.5}$$

In the above, prime denotes differentiation with respect to x_3,

$$\boldsymbol{\kappa}_{\alpha\beta} = \boldsymbol{\gamma}_{\alpha,\beta}, \qquad \boldsymbol{\kappa}_{3\alpha} = \boldsymbol{\delta}_{3,\alpha}, \qquad \boldsymbol{\gamma}_\alpha = \boldsymbol{\delta}_\alpha - H u_{3,\alpha} \boldsymbol{\sigma}, \tag{2.6}$$

and

$$\boldsymbol{\kappa}_{\alpha\beta} = \boldsymbol{\kappa}_{(\alpha\beta)} + \boldsymbol{\kappa}_{[\alpha\beta]}, \tag{2.7}$$

where

$$\boldsymbol{\kappa}_{(\alpha\beta)} = \tfrac{1}{2}(\boldsymbol{\kappa}_{\alpha\beta}+\boldsymbol{\kappa}_{\beta\alpha}), \qquad \boldsymbol{\kappa}_{[\alpha\beta]} = \tfrac{1}{2}(\boldsymbol{\kappa}_{\alpha\beta}-\boldsymbol{\kappa}_{\beta\alpha}). \tag{2.8}$$

The stress resultants and matrix stress resultants are defined, similar to [18–20], as

$$\begin{gathered} \boldsymbol{m}_{\beta\alpha} = \frac{h}{2}\int_{-1}^{1} t^*_{\beta\alpha}\boldsymbol{\zeta}\,\mathrm{d}x_3, \qquad \boldsymbol{m}_{3\alpha} = \frac{h}{2}\int_{-1}^{1} t^*_{3\alpha}\boldsymbol{\eta}\,\mathrm{d}x_3, \\ \boldsymbol{m}_{\beta} = \int_{-1}^{1} t^*_{\beta 3}\boldsymbol{\zeta}'\,\mathrm{d}x_3, \qquad \boldsymbol{m}_{3} = \int_{-1}^{1} t^*_{33}\boldsymbol{\eta}'\,\mathrm{d}x_3, \\ N_{3\alpha} = \frac{h}{2}\int_{-1}^{1} t^*_{3\alpha}\,\mathrm{d}x_3, \end{gathered} \tag{2.9}$$

and

$$\bar{\boldsymbol{m}}_{\beta} = \boldsymbol{m}_{\beta\alpha}\nu_{\alpha}, \qquad \bar{\boldsymbol{m}}_{3} = \boldsymbol{m}_{3\alpha}\nu_{\alpha}, \qquad \bar{N}_{3} = N_{3\alpha}\nu_{\alpha}, \tag{2.10}$$

where ν_α are the components of the unit outward normal to the boundary surface. The surface tractions on the planes $x_3 = \pm 1$ induce loads and matrix body couples defined by

$$p_3 = t^*_{33}|^{1}_{-1}, \qquad \boldsymbol{l}_{\beta} = t^*_{\beta 3}\boldsymbol{\zeta}|^{1}_{-1} = p_{\beta}\boldsymbol{\zeta}(1), \qquad \boldsymbol{l}_{3} = t^*_{33}\boldsymbol{\eta}|^{1}_{-1} = p_{3}\boldsymbol{\eta}(1), \tag{2.11}$$

where the boundary conditions as given by equation (2.2) have been used. In addition, since the density ρ^* is independent of x_3, the only other weighted averages that need to be introduced are

$$\begin{aligned} \rho &= \frac{h}{2}\int_{-1}^{1} \rho^*\,\mathrm{d}x_3 = \rho^* h, \qquad & \boldsymbol{k} &= \frac{1}{2}\int_{-1}^{1} \boldsymbol{\eta}\,\mathrm{d}x_3, \\ \mathfrak{R}_1 &= \frac{1}{2}\int_{-1}^{1} \boldsymbol{\zeta}\boldsymbol{\zeta}^T\,\mathrm{d}x_3, \qquad & \mathfrak{R}_2 &= \frac{1}{2}\int_{-1}^{1} \boldsymbol{\eta}\boldsymbol{\eta}^T\,\mathrm{d}x_3. \end{aligned} \tag{2.12}$$

Equations governing the stress resultants and matrix stress resultants for the bending of the plate are obtained with the aid of equations (2.4), (2.9), (2.11) and (2.12) by integrating, as well as multiplying by $\boldsymbol{\zeta}$ or $\boldsymbol{\eta}$ and integrating, the three-dimensional equations of motion. This results in the following equations

$$\begin{aligned} \frac{1}{l}N_{3\alpha,\alpha}+p_3 &= \rho\left(\frac{\partial^2 u_3}{\partial t^2}+\boldsymbol{k}^T\frac{\partial^2 \boldsymbol{\delta}_3}{\partial t^2}\right), \\ \frac{1}{l}\boldsymbol{m}_{3\alpha,\alpha}+\boldsymbol{l}_3-\boldsymbol{m}_3 &= \rho\left(\boldsymbol{k}\frac{\partial^2 u_3}{\partial t^2}+\mathfrak{R}_2\frac{\partial^2 \boldsymbol{\delta}_3}{\partial t^2}\right), \\ \frac{1}{l}\boldsymbol{m}_{\beta\alpha,\alpha}+\boldsymbol{l}_\beta-\boldsymbol{m}_\beta &= \rho\mathfrak{R}_1\frac{\partial^2 \boldsymbol{\gamma}_\beta}{\partial t^2}. \end{aligned} \tag{2.13}$$

In the above the body forces have been omitted and, of course, may easily be added. The matrix resultant $\boldsymbol{m}_\beta$ is related to the shear stress resultant $N_{3\alpha}$ as follows:

$$\begin{aligned} \boldsymbol{\sigma}^T\boldsymbol{m}_\beta &= \boldsymbol{\sigma}^T\int_{-1}^{1} t^*_{\beta 3}\boldsymbol{\zeta}'\,\mathrm{d}x_3 = \int_{-1}^{1} t^*_{\beta 3}\boldsymbol{\sigma}^T\boldsymbol{\zeta}'\,\mathrm{d}x_3 \\ &= \int_{-1}^{1} t^*_{\beta 3}\,\mathrm{d}x_3 = \int_{-1}^{1} t^*_{3\beta}\,\mathrm{d}x_3 = \frac{2}{h}N_{3\beta} \end{aligned} \tag{2.14}$$

where the derivative of equation (2.3) and some of the definitions in equation (2.9) have been used.

The constitutive equations relating the stress resultants and matrix stress resultants to the kinematic quantities defined in equation (2.4) through (2.8) may be obtained from the integrated form of the free energy (or strain energy). Since the material is assumed to be homogeneous and isotropic, the free energy of the three-dimensional theory of elasticity is given by

$$\rho^* A^* = \frac{E}{2(1+\nu)}\left\{\gamma^*_{rs}\gamma^*_{rs} + \frac{\nu}{1-2\nu}\gamma^*_{rr}\gamma^*_{ss}\right\} \qquad r, s = 1, 2, 3 \tag{2.15}$$

where E and ν are Young's modulus and Poisson's ratio respectively. The integrated form of the free energy is

$$\begin{aligned}\rho A = \frac{h}{2}\int_{-1}^{1} \rho^* A^* \,\mathrm{d}x_3 = \frac{EH}{2(1+\nu)l}\Big\{&\boldsymbol{\kappa}^T_{(\alpha\beta)}\mathfrak{B}_1\boldsymbol{\kappa}_{(\alpha\beta)} \\ &+\frac{\nu}{1-2\nu}\boldsymbol{\kappa}^T_{\alpha\alpha}\mathfrak{B}_1\boldsymbol{\kappa}_{\beta\beta} + \tfrac{1}{2}\boldsymbol{\kappa}^T_{3\alpha}\mathfrak{B}_2\boldsymbol{\kappa}_{3\alpha} + \frac{1}{H}\boldsymbol{\delta}^T_{\alpha}\mathfrak{B}_3\boldsymbol{\kappa}_{3\alpha} \\ &+\frac{2\nu}{(1-2\nu)H}\boldsymbol{\delta}^T_3\mathfrak{B}_4\boldsymbol{\kappa}_{\alpha\alpha} + \frac{1}{2H^2}\boldsymbol{\delta}^T_{\alpha}\mathfrak{B}_5\boldsymbol{\delta}_{\alpha} + \frac{1}{H^2}\left(\frac{1-\nu}{1-2\nu}\right)\boldsymbol{\delta}^T_3\mathfrak{B}_6\boldsymbol{\delta}_3\Big\},\end{aligned} \tag{2.16}$$

where $\mathfrak{B}_a$, $a = 1, 2 \ldots 6$ are $P \times P$ matrices given by

$$\begin{aligned} \mathfrak{B}_1 &= \int_{-1}^{1} \boldsymbol{\zeta}\boldsymbol{\zeta}^T \,\mathrm{d}x_3, & \mathfrak{B}_2 &= \int_{-1}^{1} \boldsymbol{\eta}\boldsymbol{\eta}^T \,\mathrm{d}x_3, \\ \mathfrak{B}_3 &= \int_{-1}^{1} \boldsymbol{\zeta}'\boldsymbol{\eta}^T \,\mathrm{d}x_3, & \mathfrak{B}_4 &= \int_{-1}^{1} \boldsymbol{\eta}'\boldsymbol{\zeta}^T \,\mathrm{d}x_3, \\ \mathfrak{B}_5 &= \int_{-1}^{1} \boldsymbol{\zeta}'(\boldsymbol{\zeta}')^T \,\mathrm{d}x_3, & \mathfrak{B}_6 &= \int_{-1}^{1} \boldsymbol{\eta}'(\boldsymbol{\eta}')^T \,\mathrm{d}x_3. \end{aligned} \tag{2.17}$$

The matrix stress resultants are then determined by using derivatives of the free energy similar to those given by Green *et al.* [18]. Thus it may be readily shown that the constitutive equations for the bending of the plate are

$$\begin{aligned} \boldsymbol{m}_{\beta\alpha} &= l\partial_{\boldsymbol{\kappa}_{\beta\alpha}}\rho A, & \boldsymbol{m}_{3\alpha} &= l\partial_{\boldsymbol{\kappa}_{3\alpha}}\rho A, \\ \boldsymbol{m}_{\beta} &= \partial_{\boldsymbol{\delta}_{\beta}}\rho A, & \boldsymbol{m}_{3} &= \partial_{\boldsymbol{\delta}_{3}}\rho A, \end{aligned} \tag{2.18}$$

where, for example, $\partial_{\boldsymbol{\delta}_3}\rho A$ is defined as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\rho A(\boldsymbol{\delta}_\beta, \boldsymbol{\delta}_3 + \tau\boldsymbol{b}, \boldsymbol{\kappa}_{\beta\alpha}, \boldsymbol{\kappa}_{3\alpha})|_{\tau=0} = (\partial_{\boldsymbol{\delta}_3}\rho A)^T\boldsymbol{b}.$$

From equations (2.16) and (2.18) the constitutive equations for the bending of elastic plates are

$$
\begin{aligned}
m_{(\beta\alpha)} &= \frac{EH}{1+\nu}\left\{\mathfrak{B}_1\kappa_{(\beta\alpha)}+\frac{\nu}{1-2\nu}\delta_{\beta\alpha}\left(\mathfrak{B}_1\kappa_{\varepsilon\varepsilon}+\frac{1}{H}\mathfrak{B}_4^T\delta_3\right)\right\},\\
m_{[\beta\alpha]} &= 0,\\
m_{3\alpha} &= \frac{EH}{2(1+\nu)}\left\{\mathfrak{B}_2\kappa_{3\alpha}+\frac{1}{H}\mathfrak{B}_3^T\delta_\alpha\right\},\\
m_\beta &= \frac{E}{2(1+\nu)l}\left\{\mathfrak{B}_3\kappa_{3\beta}+\frac{1}{H}\mathfrak{B}_5\delta_\beta\right\},\\
m_3 &= \frac{E}{(1+\nu)(1-2\nu)l}\left\{\nu\mathfrak{B}_4\kappa_{\varepsilon\varepsilon}+\frac{1-\nu}{H}\mathfrak{B}_6\delta_3\right\}.
\end{aligned}
\tag{2.19}
$$

The system of equations (2.6), (2.13), (2.14) and (2.19), with the inertia terms omitted, are equivalent to (4.5) through (4.6) given in [20] where in [20] the "material coefficients" '$\boldsymbol{\alpha}$'s' are related to the '$\mathfrak{B}$'s' and the set of functions $\{\boldsymbol{\eta}, \boldsymbol{\zeta}\}$ are taken as follows:†

$$
\zeta_N(x_3) = \left(\frac{hx_3}{2}\right)^{2N-1} = X_3^{2N-1}, \qquad \eta_N(x_3) = \left(\frac{hx_3}{2}\right)^{2N} = X_3^{2N}, \qquad 1,2,\dots P. \tag{2.20}
$$

From the definition (2.17) it follows that $\mathfrak{B}_1$, $\mathfrak{B}_2$, $\mathfrak{B}_5$ and $\mathfrak{B}_6$ are symmetric positive definite.

From the definitions (2.3), (2.12) and (2.17) the following identities are obtained

$$
\mathfrak{B}_5\boldsymbol{\sigma} = 2\boldsymbol{\zeta}(1), \qquad \boldsymbol{\sigma}^T\mathfrak{B}_5\boldsymbol{\sigma} = 2, \qquad \mathfrak{B}_3^T\boldsymbol{\sigma} = 2\boldsymbol{k}, \tag{2.21}
$$

as well as

$$
2\mathfrak{K}_1 = \mathfrak{B}_1, \qquad 2\mathfrak{K}_2 = \mathfrak{B}_2. \tag{2.22}
$$

3. DECOMPOSITION OF THE BENDING THEORY

The Stokes–Helmholtz decomposition theorem is now applied to each of the members of $\boldsymbol{\delta}_\alpha$ as well as p_α. These quantities are written as

$$
\boldsymbol{\delta}_\alpha = \boldsymbol{\varphi}_{,\alpha}+\varepsilon_{\alpha\beta}\boldsymbol{\psi}_{,\beta}, \qquad p_\alpha = f_{,\alpha}+\varepsilon_{\alpha\beta}g_{,\beta}, \tag{3.1}
$$

where $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$ are $P\times 1$ matrix functions, and f, g are scalar functions of the nondimensional plate coordinates and time. Equation (3.1), expressed in terms of $\boldsymbol{\gamma}_\alpha \equiv \boldsymbol{\delta}_\alpha - Hu_{3,\alpha}\boldsymbol{\sigma}$, is written as

$$
\boldsymbol{\gamma}_\alpha = \boldsymbol{\chi}_{,\alpha}+\varepsilon_{\alpha\beta}\boldsymbol{\psi}_{,\beta}, \tag{3.2}
$$

where

$$
\boldsymbol{\chi} = \boldsymbol{\varphi} - Hu_3\boldsymbol{\sigma}. \tag{3.3}
$$

† Note the slight change in notation on the matrix stress resultants and one of the kinematic quantities.

Substitution of equation (3.2) into equation (2.6), and using equation (2.8) results in

$$\boldsymbol{\kappa}_{(\alpha\beta)} = \boldsymbol{\chi}_{,\alpha\beta}+\varepsilon_{\alpha\nu}\boldsymbol{\psi}_{,\nu\beta}, \qquad \boldsymbol{\kappa}_{[\alpha\beta]} = \varepsilon_{\alpha\nu}\boldsymbol{\psi}_{,\nu\beta} = \tfrac{1}{2}\varepsilon_{\alpha\beta}\Delta\boldsymbol{\psi}, \tag{3.4}$$

where

$$\Delta = \frac{\partial^2}{\partial x_\alpha\,\partial x_\alpha} = \frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}, \tag{3.5}$$

is the two-dimensional Laplacian operator.

Expressions for the constitutive equations in terms of the $P\times 1$ matrix stress functions $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$, $\boldsymbol{\chi}$ and the $P\times 1$ matrix displacement $\boldsymbol{\delta}_3$ are obtained by substituting equations $(3.1)_1$, (3.4) and $(2.6)_2$ into equations (2.19) and are given by

$$\begin{aligned}
\boldsymbol{m}_{(\beta\alpha)} &= \frac{EH}{1+\nu}\left\{\mathfrak{B}_1[\boldsymbol{\chi}_{,\beta\alpha}+\varepsilon_{\beta\nu}\boldsymbol{\psi}_{,\nu\alpha}]+\frac{\nu}{1-2\nu}\delta_{\beta\alpha}\left[\mathfrak{B}_1\Delta\boldsymbol{\chi}+\frac{1}{H}\mathfrak{B}_4^T\boldsymbol{\delta}_3\right]\right\},\\
\boldsymbol{m}_{3\alpha} &= \frac{EH}{2(1+\nu)}\left\{\mathfrak{B}_2\boldsymbol{\delta}_{3,\alpha}+\frac{1}{H}\mathfrak{B}_3^T(\boldsymbol{\varphi}_{,\alpha}+\varepsilon_{\alpha\beta}\boldsymbol{\psi}_{,\beta})\right\},\\
\boldsymbol{m}_{\beta} &= \frac{E}{2(1+\nu)l}\left\{\mathfrak{B}_3\boldsymbol{\delta}_{3,\beta}+\frac{1}{H}\mathfrak{B}_5(\boldsymbol{\varphi}_{,\beta}+\varepsilon_{\beta\alpha}\boldsymbol{\psi}_{,\alpha})\right\},\\
\boldsymbol{m}_3 &= \frac{E}{(1+\nu)(1-2\nu)l}\left\{\nu\mathfrak{B}_4\Delta\boldsymbol{\chi}+\frac{1-\nu}{H}\mathfrak{B}_6\boldsymbol{\delta}_3\right\}.
\end{aligned} \tag{3.6}$$

The equations governing $\boldsymbol{\chi}$, $\boldsymbol{\psi}$, $\boldsymbol{\delta}_3$ and u_3 are obtained by using the technique outlined in Green and Naghdi [17] along with the identities given in equations (2.21) and (2.22). These results are given in terms of the velocity of dilatational waves in an unbounded medium $c_1 = [(\lambda+2\mu)/\rho^*]^{\frac{1}{2}}$ and the velocity of distortional waves in an unbounded medium $c_2 = (\mu/\rho^*)^{\frac{1}{2}}$, where λ and μ are the Lamé coefficients. Thus, following [17, 20] the set of equations governing $\boldsymbol{\chi}$, $\boldsymbol{\psi}$, $\boldsymbol{\delta}_3$ and u_3 are

$$\mathfrak{B}_1\Box_2\boldsymbol{\psi}-\frac{1}{H^2}\mathfrak{B}_5\boldsymbol{\psi}+\frac{2g}{\rho}\left(\frac{l}{c_2}\right)^2\boldsymbol{\zeta}(1) = \mathbf{0}, \tag{3.7}$$

$$\left(\frac{c_1}{c_2}\right)^2\mathfrak{B}_1\Box_1\boldsymbol{\chi}-\frac{1}{H^2}\mathfrak{B}_5\boldsymbol{\chi}+\frac{1}{H}\left\{\left[\left(\frac{c_1}{c_2}\right)^2-2\right]\mathfrak{B}_4^T-\mathfrak{B}_3\right\}\boldsymbol{\delta}_3+\frac{2}{\rho}\left(\frac{l}{c_2}\right)^2 f\boldsymbol{\zeta}(1) = \frac{2}{H}\boldsymbol{\zeta}(1)u_3, \tag{3.8}$$

$$\begin{aligned}
\mathfrak{B}_2\Box_2\boldsymbol{\delta}_3-\frac{1}{H^2}\left(\frac{c_1}{c_2}\right)^2\mathfrak{B}_6\boldsymbol{\delta}_3-\frac{1}{H}\left\{\left[\left(\frac{c_1}{c_2}\right)^2-2\right]\mathfrak{B}_4^T-\mathfrak{B}_3\right\}^T\Delta\boldsymbol{\chi}&\\
+\frac{2}{\rho}\left(\frac{l}{c_2}\right)^2 p_3\boldsymbol{\eta}(1) = -2\boldsymbol{k}\Box_2u_3&,
\end{aligned} \tag{3.9}$$

$$\frac{1}{H}\boldsymbol{\zeta}^T(1)\Delta\boldsymbol{\chi}+\boldsymbol{k}^T\Box_2\boldsymbol{\delta}_3+\frac{1}{\rho}\left(\frac{l}{c_2}\right)^2 p_3 = -\Box_2u_3, \tag{3.10}$$

where the two wave operators $\Box_1$ and $\Box_2$ are defined as

$$\Box_1 = \Delta-\left(\frac{l}{c_1}\right)^2\frac{\partial^2}{\partial t^2}, \qquad \Box_2 = \Delta-\left(\frac{l}{c_2}\right)^2\frac{\partial^2}{\partial t^2}. \tag{3.11}$$

It is noted that equation (3.7) is uncoupled from the other three equations but represents P coupled partial differential equations. Since $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are square, real and positive definite, equation (3.7) may be reduced to a set of P uncoupled equations. In addition, for the static case, the remaining three equations, viz. (3.8), (3.9) and (3.10), may also be uncoupled, and this is done in Section 5.

4. THE CANONICAL FORM OF THE EQUATIONS OF THE BENDING OF ELASTIC PLATES

Up to now the set of functions $\{1, \boldsymbol{\eta}, \boldsymbol{\zeta}\}$ have been assumed to be arbitrary except for the condition that they form a basis of the space $\mathfrak{F}(P)$. In what follows these functions are determined in such a way that the set of governing equations (3.7) through (3.10) remain invariant under a linear change of bases of the space $\mathfrak{F}(P)$. This requirement is of importance since the field equations will be the same irrespective of whether power functions or Legendre polynomials are used in the assumed form of the three-dimensional displacement field.

Since $\mathfrak{B}_1$ and $\mathfrak{B}_5$ are square, real and positive definite, as well as $\mathfrak{B}_2$ and $\mathfrak{B}_6$, the standard reduction of simultaneous quadratic forms may be used (see, e.g. Mirsky [21, Theorem 13.4.2]). The following notation is now introduced in the reduction of the pairs of simultaneous quadratic forms

$$
\begin{aligned}
\mathfrak{P}_1^T\mathfrak{B}_1\mathfrak{P}_1 &= \mathfrak{I}, \qquad \mathfrak{P}_1^T\mathfrak{B}_5\mathfrak{P}_1 = \mathfrak{D}_1 = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_P), \\
\mathfrak{P}_2^T\mathfrak{B}_2\mathfrak{P}_2 &= \mathfrak{I}, \qquad \mathfrak{P}_2^T\mathfrak{B}_6\mathfrak{P}_2 = \mathfrak{D}_2 = \operatorname{diag}(\mu_1, \mu_2, \ldots \mu_P),
\end{aligned}
\tag{4.1}
$$

where $\lambda_1, \lambda_2, \ldots \lambda_P$ and $\mu_1, \mu_2, \ldots \mu_P$ are the roots of the equations

$$
\det(\mathfrak{B}_5 - \lambda\mathfrak{B}_1) = 0, \qquad \det(\mathfrak{B}_6 - \mu\mathfrak{B}_2) = 0,
\tag{4.2}
$$

respectively. In addition $\mathfrak{I}$ is the $P \times P$ identity matrix, and the $P \times P$ matrices $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are

$$
\mathfrak{P}_1 = \begin{pmatrix} | & | & & | \\ \boldsymbol{u}_1 & \boldsymbol{u}_2 & \ldots & \boldsymbol{u}_P \\ | & | & & | \end{pmatrix}, \qquad \mathfrak{P}_2 = \begin{pmatrix} | & | & & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \ldots & \boldsymbol{v}_P \\ | & | & & | \end{pmatrix},
\tag{4.3}
$$

where $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots \boldsymbol{u}_P$ and $\boldsymbol{v}_1, \boldsymbol{v}_2 \ldots \boldsymbol{v}_P$ are the normalized eigenvectors determined by

$$
\begin{aligned}
(\mathfrak{B}_5 - \lambda_r\mathfrak{B}_1)\bar{\boldsymbol{u}}_r &= 0, \qquad \boldsymbol{u}_r = \frac{\bar{\boldsymbol{u}}_r}{(\bar{\boldsymbol{u}}_r^T\mathfrak{B}_1\bar{\boldsymbol{u}}_r)^{\frac{1}{2}}}, \\
(\mathfrak{B}_6 - \mu_r\mathfrak{B}_2)\bar{\boldsymbol{v}}_r &= 0, \qquad \boldsymbol{v}_r = \frac{\bar{\boldsymbol{v}}_r}{(\bar{\boldsymbol{v}}_r^T\mathfrak{B}_2\bar{\boldsymbol{v}}_r)^{\frac{1}{2}}}.
\end{aligned}
\tag{4.4}
$$

The above reduction of simultaneous quadratic forms introduces a change of bases which is denoted by

$$
\mathbf{Z} = \mathfrak{P}_1^T\boldsymbol{\zeta}, \qquad \mathbf{H} = \mathfrak{P}_2^T\boldsymbol{\eta}.
\tag{4.5}
$$

Thus, with the introduction of the following quantities

$$
\begin{aligned}
&\boldsymbol{\psi} = \mathfrak{P}_1\boldsymbol{\Psi}, \quad &&\boldsymbol{\chi} = \mathfrak{P}_1\mathbf{X}, \quad &&\boldsymbol{\varphi} = \mathfrak{P}_1\boldsymbol{\Phi}, \quad &&\boldsymbol{\sigma} = \mathfrak{P}_1\boldsymbol{\Sigma},\\
&\boldsymbol{\delta} = \mathfrak{P}_2\boldsymbol{\Delta}_3, \quad &&\boldsymbol{K} = \mathfrak{P}_2^T\boldsymbol{k}, \quad &&\mathfrak{C}_1 = \mathfrak{P}_1^T\mathfrak{B}_3\mathfrak{P}_2, \quad &&\mathfrak{C}_2 = \mathfrak{P}_2^T\mathfrak{B}_4\mathfrak{P}_1,
\end{aligned}
\tag{4.6}
$$

as well as

$$
\mathfrak{C} = \frac{2\nu}{1-2\nu}\mathfrak{C}_2^T - \mathfrak{C}_1 = \left[\left(\frac{c_1}{c_2}\right)^2 - 2\right]\mathfrak{C}_2^T - \mathfrak{C}_1,
\tag{4.7}
$$

equations (3.7)–(3.10) may be written in the following canonical form

$$
\left[\mathfrak{J}\,\Box_2 - \frac{1}{H^2}\mathfrak{D}_1\right]\boldsymbol{\Psi} + \frac{2}{\rho}\left(\frac{l}{c_2}\right)^2 g\mathbf{Z}(1) = \mathbf{0},
\tag{4.8}
$$

$$
\left[\mathfrak{J}\left(\frac{c_1}{c_2}\right)^2\Box_1 - \frac{1}{H^2}\mathfrak{D}_1\right]\mathbf{X} + \frac{1}{H}\mathfrak{C}\boldsymbol{\Delta}_3 + \frac{2}{\rho}\left(\frac{l}{c_2}\right)^2 f\boldsymbol{Z}(1) = \frac{2}{H}\boldsymbol{Z}(1)u_3,
\tag{4.9}
$$

$$
\left[\mathfrak{J}\,\Box_2 - \frac{1}{H^2}\left(\frac{c_1}{c_2}\right)^2\mathfrak{D}_2\right]\boldsymbol{\Delta}_3 - \frac{1}{H}\mathfrak{C}^T\Delta\mathbf{X} + \frac{2}{\rho}\left(\frac{l}{c_2}\right)^2 p_3\mathbf{H}(1) = -2\boldsymbol{K}\Box_2 u_3,
\tag{4.10}
$$

$$
\frac{1}{H}\mathbf{Z}^T(1)\Delta\mathbf{X} + \boldsymbol{K}^T\Box_2\boldsymbol{\Delta}_3 + \frac{1}{\rho}\left(\frac{l}{c_2}\right)^2 p_3 = -\Box_2 u_3,
\tag{4.11}
$$

and $\boldsymbol{\Phi}$ is determined by

$$
\boldsymbol{\Phi} = \mathbf{X} + Hu_3\boldsymbol{\Sigma}.
\tag{4.12}
$$

Equation (4.8) is a set of P uncoupled partial differential equations, and they are related to the propagation of shear waves since it involves only the distortional waves in an unbounded medium $c_2 = (\mu/\rho^*)^{\frac{1}{2}}$. Equations (4.9) and (4.10) are $2P$ coupled partial differential equations which are coupled to the scalar partial differential equation (4.11). The independent variables in these equations are the nondimensional plate coordinates and time, since the thickness coordinate has been eliminated through the introduction of the set of functions $\boldsymbol{\Upsilon}$.

The set $\{1, \mathbf{H}, \boldsymbol{Z}\}$ will be called the canonical bases and is related to the original set $\{1, \boldsymbol{\eta}, \boldsymbol{\zeta}\}$ by equation (4.5). It is useful to reformulate the results in terms of the canonical bases. From equations (2.17), (4.1), $(4.6)_7$, $(4.6)_8$ and (4.5) it follows that

$$
\begin{aligned}
&\mathfrak{J} = \int_{-1}^{1}\mathbf{Z}\mathbf{Z}^T\,\mathrm{d}x_3, \quad &&\mathfrak{J} = \int_{-1}^{1}\mathbf{H}\mathbf{H}^T\,\mathrm{d}x_3, \quad &&\mathfrak{C}_1 = \int_{-1}^{1}\mathbf{Z}'\mathbf{H}^T\,\mathrm{d}x_3,\\
&\mathfrak{C}_2 = \int_{-1}^{1}\mathbf{H}'\mathbf{Z}^T\,\mathrm{d}x_3, \quad &&\mathfrak{D}_1 = \int_{-1}^{1}\mathbf{Z}'(\mathbf{Z}')^T\,\mathrm{d}x_3, \quad &&\mathfrak{D}_2 = \int_{-1}^{1}\mathbf{H}'(\mathbf{H}')^T\,\mathrm{d}x_3.
\end{aligned}
\tag{4.13}
$$

The canonical matrix stress resultants, which are obtained from equations (2.9), (3.6), (4.1), (4.5) and (4.6), are denoted by

$$\begin{aligned}
M_{\beta\alpha} &\equiv \mathfrak{P}_1^T m_{\beta\alpha} = \frac{EH}{1+\nu}\left\{\mathbf{X}_{,\beta\alpha}+\varepsilon_{\beta\nu}\mathbf{\Psi}_{,\nu\alpha}+\frac{\nu}{1-2\nu}\delta_{\beta\alpha}\left[\Delta\mathbf{X}+\frac{1}{H}\mathfrak{C}_2^T\mathbf{\Delta}_3\right]\right\},\\
M_{3\alpha} &\equiv \mathfrak{P}_2^T m_{3\alpha} = \frac{EH}{2(1+\nu)}\left\{\mathbf{\Delta}_{3,\alpha}+\frac{1}{H}\mathfrak{C}_1^T(\mathbf{\Phi}_{,\alpha}+\varepsilon_{\alpha\beta}\mathbf{\Psi}_{,\beta})\right\},\\
M_{\beta} &\equiv \mathfrak{P}_1^T m_{\beta} = \frac{E}{2(1+\nu)l}\left\{\mathfrak{C}_1\mathbf{\Delta}_{3,\beta}+\frac{1}{H}\mathfrak{D}_1(\mathbf{\Phi}_{,\beta}+\varepsilon_{\beta\alpha}\mathbf{\Psi}_{,\alpha})\right\},\\
M_{3} &\equiv \mathfrak{P}_2^T m_{3} = \frac{E}{(1+\nu)(1-2\nu)l}\left\{\nu\mathfrak{C}_2\Delta\mathbf{X}+\frac{1-\nu}{H}\mathfrak{D}_2\mathbf{\Delta}_3\right\}.
\end{aligned} \tag{4.14}$$

The three-dimensional displacement field is determined, with the aid of equations (2.4), (3.2), (4.5) and (4.6), as

$$\begin{aligned}
u_\alpha^* &= \mathbf{Z}^T(\mathbf{X}_{,\alpha}+\varepsilon_{\alpha\beta}\mathbf{\Psi}_{,\beta}),\\
u_3^* &= u_3+\mathbf{H}^T\mathbf{\Delta}_3.
\end{aligned} \tag{4.15}$$

Finally, the boundary conditions for the edge of the plate are

$$\begin{aligned}
&\text{either } M_{\alpha\beta}\nu_\beta \text{ specified or } \mathbf{\Gamma}_\alpha \text{ specified}\\
&\text{either } M_{3\alpha}\nu_\alpha \text{ specified or } \mathbf{\Delta}_3 \text{ specified}\\
&\text{either } N_{3\alpha}\nu_\alpha \text{ specified or } u_3 \text{ specified}
\end{aligned} \tag{4.16}$$

where

$$\mathbf{\Gamma}_\alpha = \mathfrak{P}_1^{-1}\gamma_\alpha.$$

The nomenclature is such that the canonical form is obtained by replacing minuscles by majuscules except for the "material coefficients" (2.17) which are replaced by the conditions (4.13).

The following useful identities are readily shown from equations (2.21), (4.1), (4.5) and (4.6):

$$\mathfrak{D}_1\mathbf{\Sigma} = 2\mathbf{Z}(1), \qquad \mathbf{\Sigma}^T\mathfrak{D}_1\mathbf{\Sigma} = 2, \qquad \mathfrak{C}_1^T\mathbf{\Sigma} = 2\boldsymbol{K} \tag{4.17}$$

as well as

$$\mathfrak{C}_1+\mathfrak{C}_2^T = 2\mathbf{Z}(1)\mathbf{H}^T(1), \qquad \mathbf{\Sigma}^T\mathbf{Z}(1) = 1, \qquad \boldsymbol{K} = \mathfrak{C}_1^T\mathfrak{D}_1^{-1}\mathbf{Z}(1). \tag{4.18}$$

The canonical bases are now selected since the set of governing equations (4.8)–(4.11) remain invariant under a linear transformation of the bases $\mathbf{\Upsilon} = \{1, \boldsymbol{\eta}, \boldsymbol{\zeta}\}$. In order to show this consider another set of functions $\mathbf{\Upsilon}^+ = \{1, \boldsymbol{\eta}^+, \boldsymbol{\zeta}^+\}$ which are related to $\mathbf{\Upsilon}$ by the linear transformation

$$\boldsymbol{\zeta}^+ = \mathfrak{Q}_1\boldsymbol{\zeta}, \qquad \boldsymbol{\eta}^+ = \mathfrak{Q}_2\boldsymbol{\eta}, \tag{4.19}$$

or written out

$$\zeta_N^+(x_3) = \sum_{M=1}^{P}\mathfrak{Q}_{1NM}\zeta_M(x_3), \qquad \eta_N^+(x_3) = \sum_{M=1}^{P}\mathfrak{Q}_{2NM}\eta_M(x_3),$$

where $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ are $P \times P$ nonsingular matrices. The conditions on the reduction of the simultaneous quadratic forms require that

$$\mathbf{Z}^+ = \mathbf{Z}, \qquad \mathbf{H}^+ = \mathbf{H}, \tag{4.20}$$

so that equations (4.13) remain invariant under the change of bases (4.19) [i.e., $\mathfrak{D}_\mathfrak{a}^+ = \mathfrak{D}_\mathfrak{a}$, $\mathfrak{C}_\mathfrak{a}^+ = \mathfrak{C}_\mathfrak{a}$, $\mathfrak{a} = 1, 2$ and thus from equation (4.7) $\mathfrak{C}^+ = \mathfrak{C}$]. In addition the new set of eigenvectors are related to the old set (4.3) and (4.4) by

$$\mathfrak{P}_1^+ = (\mathfrak{Q}_1^T)^{-1}\mathfrak{P}_1, \qquad \mathfrak{P}_2^+ = (\mathfrak{Q}_2^T)^{-1}\mathfrak{P}_2. \tag{4.21}$$

Since u_α^*, u_3^* and x_3 are unchanged under the change of bases (4.19), it follows from equations (2.3), (2.4) and (3.2) that

$$\boldsymbol{\psi}^+ = (\mathfrak{Q}_1^T)^{-1}\boldsymbol{\psi}, \qquad \boldsymbol{\chi}^+ = (\mathfrak{Q}_1^T)^{-1}\boldsymbol{\chi}, \qquad \boldsymbol{\delta}_3 = (\mathfrak{Q}_2^T)^{-1}\boldsymbol{\delta}_3, \tag{4.22}$$

and from equations $(2.12)_2$ and $(4.19)_2$ requires that

$$\boldsymbol{k}^+ = \mathfrak{Q}_2\boldsymbol{k}. \tag{4.23}$$

Substitution of equations (4.21)–(4.23) into the appropriate expressions in (4.6) yields

$$\boldsymbol{\Psi}^+ = \boldsymbol{\Psi}, \qquad \mathbf{X}^+ = \mathbf{X}, \qquad \boldsymbol{\Sigma}^+ = \boldsymbol{\Sigma}, \qquad \boldsymbol{\Delta}_3^+ = \boldsymbol{\Delta}_3, \qquad \mathbf{K}^+ = \mathbf{K}. \tag{4.24}$$

Hence, the canonical equations (4.8)–(4.11) are invariant under a change of bases (4.19). The canonical matrix stress resultants (4.14) are also invariant under (4.19) due to (2.9) and (4.20). It should be mentioned that equations (4.8)–(4.11) are not invariant under the transformation

$$\boldsymbol{\eta}^+ = \boldsymbol{\eta} + \boldsymbol{b}, \tag{4.25}$$

or written out

$$\eta_N^+(x_3) = \eta_N(x_3) + b_N, \qquad N = 1, 2, \dots P$$

where b_N are a set of constants. It may be shown, however, that equations (3.7)–(3.10) are invariant under the transformation (4.25). The selection of $\boldsymbol{b}$ changes the functions $\mathbf{H}$, as well as the physical meaning of u_3. The functions $\boldsymbol{\zeta}$ are odd and continuous and thus $\boldsymbol{\zeta}(0) = \mathbf{0}$. No similar requirement is imposed on the functions $\boldsymbol{\eta}$ so that the functions $\boldsymbol{\eta}$ are now selected so that $\boldsymbol{\eta}(1) = \mathbf{0}$. Thus from (2.4) u_3 represents the transverse displacement on the upper or lower surfaces.

For a given value of P ($P \geq 2$) (since $P = 1$ is a degenerate case) the set $\{\mathbf{H}, \mathbf{Z}\}$ may readily be calculated by numerical means on a computer. In what follows only selected results are recorded. For the case when $P = 2$ the eigenvalues are

$$\begin{aligned} \lambda_1 &= 2{\cdot}4688, & \lambda_2 &= 42{\cdot}5312 \\ \mu_1 &= 2{\cdot}4674, & \mu_2 &= 25{\cdot}5389, \end{aligned} \tag{4.26}$$

and

$$\begin{aligned} \mathbf{Z}(x_3) &= \begin{pmatrix} 1{\cdot}5397x_3 - 0{\cdot}5386x_3^3 \\ 2{\cdot}6465x_3 - 4{\cdot}6460x_3^3 \end{pmatrix} \\ \mathbf{H}(x_3) &= \begin{pmatrix} 0{\cdot}9991 - 1{\cdot}2198x_3^2 + 0{\cdot}2207x_3^4 \\ 0{\cdot}8016 - 6{\cdot}6672x_3^2 + 5{\cdot}8656x_3^4 \end{pmatrix}. \end{aligned} \tag{4.27}$$

The graphs of these functions are shown in Figs. 1 and 2. For all practical purposes the first transverse mode shapes are approximately $\sin(\pi/2)x_3$ and $\cos(\pi/2)x_3$ as well as $\lambda_1 \doteq \mu_1 \doteq (\pi/2)^2 = 2{\cdot}4674$. The second transverse mode shapes differ from the exact transverse mode shapes, but improvement is obtained as P increases. This result is shown in Figs. 3 and 4. These figures are only for the odd functions $Z_N(x_3)$ $N = 1, 2, \ldots P$ for values of P equal to 3 and 4. Similar results also apply to the even functions $H_N(x_3)$, $N = 1, \ldots P$, where it is understood that the shape of these functions depends upon the

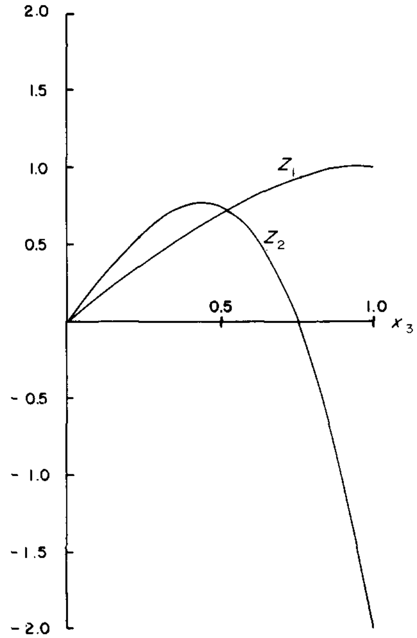

FIG. 1. Odd canonical thickness expansion functions for $P = 2$.

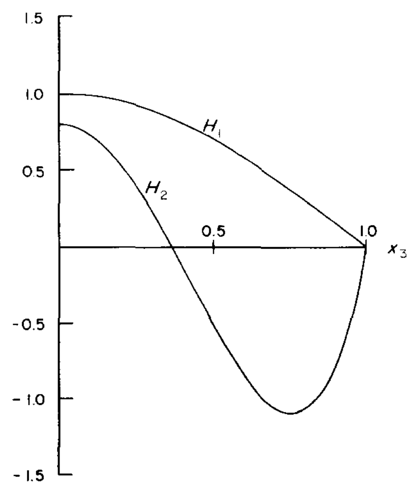

FIG. 2. Even canonical thickness expansion functions for $P = 2$.

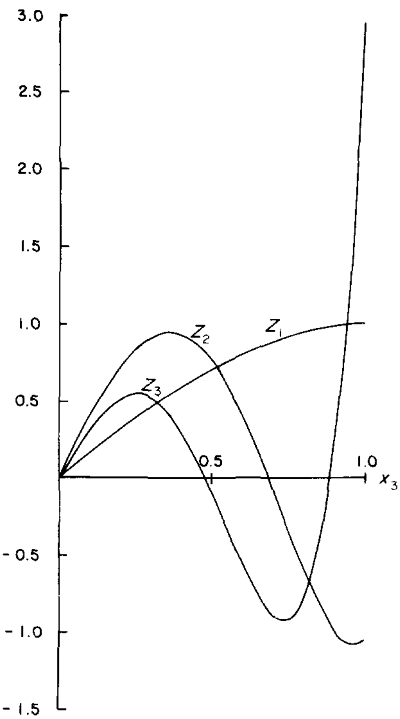

FIG. 3. Odd canonical thickness expansion functions for $P = 3$.

physical interpretation of u_3 which has been selected to represent the transverse displacement of the upper or lower surfaces. It is not surprising that the canonical functions shown in Figs. 1, 3 and 4 approximate the thickness mode shapes since they are obtained by solving eigenvalue problems given by equations (4.1)–(4.4). The matrices $\mathfrak{B}_\alpha$, $\alpha = 1, 2 \ldots 6$ are all the possible products of the set of functions $\boldsymbol{\zeta} = \{\zeta_1, \zeta_2, \ldots \zeta_P\}$ and $\boldsymbol{\eta} = \{\eta_1, \eta_2, \ldots \eta_P\}$ integrated over the thickness of the plate. Thus we are finding approximate eigenfunctions in the thickness direction when P is a finite integer. The method of finding the canonical set $\mathbf{Z} = \{Z_1, Z_2, \ldots Z_P\}$ and $\mathbf{H} = (H_1, H_2, \ldots H_P\}$ is to select a set of functions which forms a basis of the space $\mathfrak{F}_{(P)}$, for example if $P = 2$ one may select $\zeta_1(x_3) = x_3$, $\zeta_2(x_3) = x_3^3$, $\eta_1(x_3) = 1 - x_3^2$, $\eta_2(x_3) = 1 - x_3^4$. With these functions for $\boldsymbol{\zeta}$, $\boldsymbol{\eta}$ calculate $\mathfrak{B}_\alpha$, $\alpha = 1, 2 \ldots 6$, given by equation (2.17) and then simultaneously diagonize $\mathbf{B}_1$ and $\mathbf{B}_5$ as well as $\mathfrak{B}_2$ and $\mathfrak{B}_6$. The selection of ζ_1, ζ_2, η_1 and η_2 given above does not affect the results due to equations (4.19)–(4.24). For a fixed value of P the system of equations (4.8)–(4.11) is closed and the error introduced depends only on the integer P. Thus equations (4.8)–(4.11) for a given value of P is an approximate theory of the bending of elastic plates, the error only depends upon the order of the approximation and the error should decrease as P increases. The approximate constitutive equations are given by equation (4.14), where $\mathfrak{D}_\alpha$ and $\mathfrak{C}_\alpha$, $\alpha = 1, 2$, are determined from equation (4.13). For example with $P = 2$, $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are diagonal 2×2 matrices with λ_1, λ_2 and μ_1, μ_2, as given in equation (4.26), as the diagonal elements respectively. Also $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are obtained from equations $(4.13)_3$, $(4.13)_4$ and (4.47) by integration.

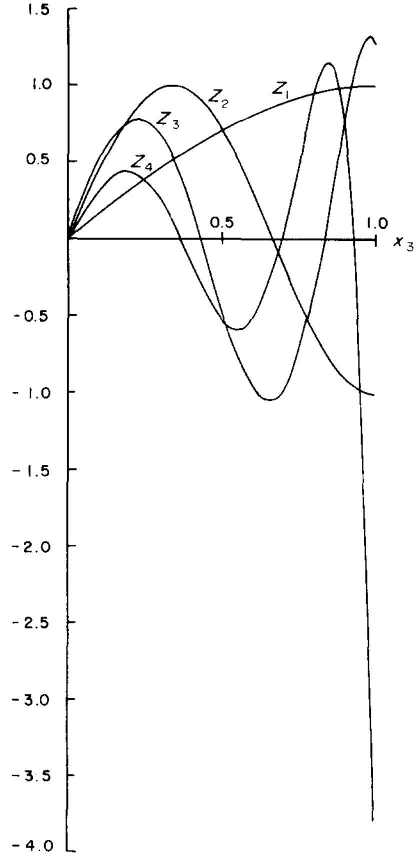

FIG. 4. Odd canonical thickness expansion functions for $P = 4$.

When P tends to infinity a set of functions Υ which makes the set of equations (3.7)–(3.10) into the canonical form (4.8)–(4.11) are

$$Z_N(x_3) = \sin\frac{(2N-1)}{2}\pi x_3, \qquad H_N(x_3) = \cos\frac{(2N-1)}{2}\pi x_3. \tag{4.28}$$

It follows from equations (4.13) and (4.28) that

$$\mathfrak{D}_1 = \mathfrak{D}_2 = \left(\frac{\pi}{2}\right)^2 \mathfrak{D}^2, \qquad \mathfrak{C}_1 = -\mathfrak{C}_2 = \frac{\pi}{2}\mathfrak{D}, \tag{4.29}$$

where

$$\mathfrak{D}_{NM} = (2N-1)\delta_{NM} \qquad N, M = 1, 2 \ldots. \tag{4.30}$$

Also from equations (4.7), $(4.17)_1$, $(4.18)_3$ and (4.29) follows

$$\mathfrak{C} = \left[1-\left(\frac{c_1}{c_2}\right)^2\right]\frac{\pi}{2}\mathfrak{D}, \qquad \Sigma = \frac{8}{\pi^2}\mathfrak{D}^{-2}\mathbf{Z}(1), \qquad K = \frac{2}{\pi}\mathfrak{D}^{-1}\mathbf{Z}(1). \tag{4.31}$$

Substitution of the values given in equations (4.29) and (4.31) into the basic equations (4.8)–(4.11) which govern $\boldsymbol{\Psi}$, $\mathbf{X}$, $\boldsymbol{\Delta}_3$ and u_3 results in

$$\left[\mathfrak{J}\square_2-\left(\frac{\pi}{2H}\right)^2\mathfrak{D}^2\right]\boldsymbol{\Psi}+\frac{2g}{\rho}\left(\frac{l}{c_2}\right)^2\mathbf{Z}(1)=\mathbf{0} \tag{4.32}$$

$$\left[\mathfrak{J}\square_1-\left(\frac{\pi}{2H}\right)^2\left(\frac{c_2}{c_1}\right)^2\mathfrak{D}^2\right]\mathbf{X}-\frac{\pi}{2H}\left[1-\left(\frac{c_2}{c_1}\right)^2\right]\mathfrak{D}\boldsymbol{\Delta}_3+\frac{2}{\rho}\left(\frac{l}{c_1}\right)^2 f\mathbf{Z}(1)=\frac{2}{H}\left(\frac{c_2}{c_1}\right)^2\mathbf{Z}(1)u_3, \tag{4.33}$$

$$\left[\mathfrak{J}\square_2-\left(\frac{\pi}{2H}\right)^2\left(\frac{c_1}{c_2}\right)^2\mathfrak{D}^2\right]\boldsymbol{\Delta}_3-\frac{\pi}{2H}\left[1-\left(\frac{c_1}{c_2}\right)^2\right]\mathfrak{D}\Delta\mathbf{X}=-\frac{4}{\pi}\mathfrak{D}^{-1}\mathbf{Z}(1)\square_2 u_3, \tag{4.34}$$

$$\mathbf{Z}^T(1)\left\{\Delta\mathbf{X}+\frac{2H}{\pi}\mathfrak{D}^{-1}\square_2\boldsymbol{\Delta}_3\right\}+\frac{H}{\rho}\left(\frac{l}{c_2}\right)^2 p_3=-H\square_2 u_3, \tag{4.35}$$

and the stress resultants from equation (4.14) are determined by

$$\begin{aligned}
\boldsymbol{M}_{\beta\alpha}&=\frac{EH}{1+\nu}\left\{\mathbf{X}_{,\beta\alpha}+\varepsilon_{\beta\nu}\boldsymbol{\Psi}_{,\nu\alpha}+\frac{\nu}{1-2\nu}\delta_{\beta\alpha}\left(\Delta X-\frac{\pi}{2H}\mathfrak{D}\boldsymbol{\Delta}_3\right)\right\},\\
\boldsymbol{M}_{3\alpha}&=\frac{EH}{2(1+\nu)}\left\{\boldsymbol{\Delta}_{3,\alpha}+\frac{\pi}{2H}\mathfrak{D}(\boldsymbol{\Phi}_{,\alpha}+\varepsilon_{\alpha\beta}\boldsymbol{\Psi}_{,\beta})\right\},\\
\boldsymbol{M}_{\beta}&=\frac{E\pi}{4(1+\nu)l}\mathfrak{D}\left\{\boldsymbol{\Delta}_{3,\beta}+\frac{\pi}{2H}\mathfrak{D}(\boldsymbol{\Phi}_{,\beta}+\varepsilon_{\beta\alpha}\boldsymbol{\Psi}_{,\alpha})\right\},\\
\boldsymbol{M}_3&=\frac{E\pi\mathfrak{D}}{2(1+\nu)(1-2\nu)l}\left\{-\nu\Delta\mathbf{X}+(1-\nu)\left(\frac{\pi}{2H}\right)\mathfrak{D}\boldsymbol{\Delta}_3\right\},
\end{aligned}\tag{4.36}$$

where

$$\boldsymbol{\Phi}=\mathbf{X}+\frac{8H}{\pi^2}\mathfrak{D}^{-2}\mathbf{Z}(1)u_3. \tag{4.37}$$

5. STATIC CASE WITH TRANSVERSE LOADS

Attention is now restricted to the equilibrium of an elastic plate subjected only to normal surface tractions. These restrictions will simplify the further reduction of the equations governing $\mathbf{X}$, $\boldsymbol{\Delta}_3$ and u_3 and will allow a comparison with the results given by Luré. With the above restrictions equations (4.32)–(4.35) reduce to

$$\left[\mathfrak{J}\Delta-\left(\frac{\pi}{2H}\right)^2\mathfrak{D}^2\right]\boldsymbol{\Psi}=\mathbf{0}, \tag{5.1}$$

$$\left\{\mathfrak{J}\Delta-\left[\frac{1-2\nu}{2(1-\nu)}\right]\left(\frac{\pi}{2H}\right)^2\mathfrak{D}^2\right\}\mathbf{X}-\frac{1}{2(1-\nu)}\frac{\pi}{2H}\mathfrak{D}\boldsymbol{\Delta}_3=\left[\frac{1-2\nu}{2(1-\nu)}\right]\frac{2}{H}\mathbf{Z}(1)u_3, \tag{5.2}$$

$$\left\{\mathfrak{J}\Delta-\left[\frac{2(1-\nu)}{1-2\nu}\right]\left(\frac{\pi}{2H}\right)^2\mathfrak{D}^2\right\}\boldsymbol{\Delta}_3+\frac{1}{(1-2\nu)}\frac{\pi}{2H}\mathfrak{D}\Delta\mathbf{X}=-\frac{4}{\pi}\mathfrak{D}^{-1}\mathbf{Z}(1)u_3, \tag{5.3}$$

$$\mathbf{Z}^T(1)\left\{\Delta\mathbf{X}+\frac{2H}{\pi}\mathfrak{D}^{-1}\Delta\boldsymbol{\Delta}_3\right\}+\frac{l(1+\nu)}{E}p_3=-H\Delta u_3. \tag{5.4}$$

In order to obtain an equation governing u_3 first solve for Δ_3 from equation (5.2) so that

$$\Delta_3 = 2(1-\nu)\left(\frac{\pi}{2H}\right)^{-1}\mathfrak{D}^{-1}\left\{\mathfrak{I}\,\Delta-\left[\frac{1-2\nu}{2(1-\nu)}\right]\left(\frac{\pi}{2H}\right)^2\mathfrak{D}^2\right\}\mathbf{X}-\frac{4}{\pi}(1-2\nu)\mathfrak{D}^{-1}\mathbf{Z}(1)u_3. \quad (5.5)$$

Next substitute equation (5.5) into (5.3) as well as (5.4), and use the fact that

$$\mathbf{Z}^T(1)\mathfrak{D}^{-2n}\mathbf{Z}(1) = \sum_{N=1}^{\infty}\frac{1}{(2N-1)^{2n}} = \frac{(-1)^{n+1}2^{2n-1}(2^{2n}-1)}{(2n)!}\left(\frac{\pi}{2}\right)^{2n}B_{2n}, \quad (5.6)$$

where B_{2n} are the Bernoulli numbers [e.g., 22] defined by

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty}\frac{B_n x^n}{n!} \qquad |x| < 2\pi, \quad (5.7)$$

to obtain

$$\left[\mathfrak{I}\,\Delta-\left(\frac{\pi}{2H}\right)^2\mathfrak{D}^2\right]^2 X = -\frac{1}{H}\frac{2\nu}{(1-\nu)}\left[\mathfrak{I}\,\Delta+\left(\frac{1-\nu}{\nu}\right)\left(\frac{\pi}{2H}\right)^2\mathfrak{D}^2\right]\mathbf{Z}(1)u_3, \quad (5.8)$$

$$\mathbf{Z}^T(1)\left\{\left(\frac{\pi}{2H}\right)^{-2}\mathfrak{D}^{-2}\left[\mathfrak{I}\,\Delta+\frac{\nu}{1-\nu}\left(\frac{\pi}{2H}\right)^2\mathfrak{D}^2\right]\Delta X\right\}+\frac{l}{2E}\left(\frac{1+\nu}{1-\nu}\right)p_3 = \frac{-\nu}{1-\nu}H\Delta u_3. \quad (5.9)$$

From equation (5.8) solve for $\mathbf{X}$ which results in

$$\mathbf{X} = -\frac{2}{H}\left\{\sum_{n=1}^{\infty}\frac{n-\nu}{1-\nu}\left(\frac{\pi}{2H}\mathfrak{D}\right)^{-2n}\Delta^{n-1}\right\}\mathbf{Z}(1)u_3, \quad (5.10)$$

and upon substitution of equation (5.10) into (5.9) finally gives the partial differential equation governing u_3 as

$$\frac{3}{2H^4}\sum_{n=1}^{\infty}\frac{(-1)^n n}{[2(n+1)]!}2^{2n+3}(2^{2(n+1)}-1)B_{2(n+1)}H^{2(n+1)}\Delta^{n+1}u_3 = \frac{l^4 p_3}{D}, \quad (5.11)$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)} = \frac{2}{3}l^3\frac{EH^3}{1-\nu^2}. \quad (5.12)$$

Equation (5.11) written out is

$$\Delta^2\{1+\tfrac{4}{5}H^2\Delta+\tfrac{17}{35}H^4\Delta^2+\tfrac{248}{945}H^6\Delta^3+\ldots\}u_3 = \frac{l^4 p_3}{D}. \quad (5.13)$$

Thus the bending of an elastic layer bounded by two parallel planes with normal loads $\frac{1}{2}p_3$ on the upper surface and $\frac{1}{2}p_3$ on the lower surface has been reduced to the solution of equations (5.13) and (5.1) which when written out is

$$\Delta\Psi_N-\left[\frac{\pi}{2H}(2N-1)\right]^2\Psi_N = 0, \qquad N = 1,2,\ldots. \quad (5.14)$$

Introduction of the concept of functions of operators defined in their usual way [e.g. 23] allows the following information for u_3. Now since

$$\sec^2 x - \frac{\tan x}{x} = \sum_{n=1}^{\infty} (-1)^n \frac{n}{[2(n+1)]!} 2^{2n+3}(2^{2(n+1)}-1)B_{2(n+1)}x^{2n}, \tag{5.15}$$

equation (5.11) may be written as

$$\frac{3}{2H^4}\{\sec^2 H\sqrt{\Delta} - (H\sqrt{\Delta})^{-1} \tan H\sqrt{\Delta}\}(H\sqrt{\Delta})^2 u_3 = \frac{l^4 p_3}{D}, \tag{5.16}$$

or equivalently

$$\frac{3}{2H^4} \sec^2 H\sqrt{\Delta}\left\{1 - \frac{\sin 2H\sqrt{\Delta}}{2H\sqrt{\Delta}}\right\}(H\sqrt{\Delta})^2 u_3 = \frac{l^4 p_3}{D}. \tag{5.17}$$

Expressions (5.16) and (5.17) are equivalent to (5.11) where, for example, the notation $\sin 2H\sqrt{\Delta}/2H\sqrt{\Delta}$ means

$$\frac{\sin 2H\sqrt{\Delta}}{2H\sqrt{\Delta}} \equiv 1 - \frac{(2H)^2\Delta}{3!} + \frac{(2H)^2\Delta^2}{5!} - \ldots. \tag{5.18}$$

In order to express the above results in a formulation similar to Luré the scalar stress function ψ, defined by

$$\psi = \frac{1}{2(1-\nu)} \sec^2(H\sqrt{\Delta})u_3, \tag{5.19}$$

is introduced. Then from (5.17) ψ must satisfy

$$\left(1 - \frac{\sin 2H\sqrt{\Delta}}{2H\sqrt{\Delta}}\right)\Delta\psi = \frac{p_3}{2lHG}, \tag{5.20}$$

where G is the shear modulus. Equations (5.20) is equivalent to equation (3.3.30) of A. I. Luré [2], and (5.14) is given in the summary of the results of Luré in [3–5]. It should be mentioned that the method of solution is not the same as that given by Luré even though it is possible to obtain all of his results from the above analysis.

In summary the method of solution consists first in solving the partial differential equations (5.13) and (5.14). Then $\mathbf{X}$, $\boldsymbol{\Delta}_3$ and $\boldsymbol{\Phi}$ may be determined with the aid of equations (5.10), (5.5) and (4.37), which are expressed in terms of $\boldsymbol{\Psi}$ and u_3 as follows:

$$\mathbf{X} = -\frac{2}{H}\left\{\sum_{n=1}^{\infty} \frac{n-\nu}{1-\nu}\left(\frac{\pi}{2H}\mathfrak{D}\right)^{-2n} \Delta^{n-1}u_3\right\} \mathbf{Z}(1), \tag{5.21}$$

$$\boldsymbol{\Delta}_3 = \frac{2}{H}\left\{\sum_{n=1}^{\infty} \left(1 - \frac{n}{1-\nu}\right)\left(\frac{\pi}{2H}\mathfrak{D}\right)^{-2n-1} \Delta^n u_3\right\} \mathbf{Z}(1), \tag{5.22}$$

$$\boldsymbol{\Phi} = \frac{-2}{H}\left\{\sum_{n=1}^{\infty} \left(1 + \frac{n}{1-\nu}\right)\left(\frac{\pi}{2H}\mathfrak{D}\right)^{-2n-2} \Delta^n u_3\right\} \mathbf{Z}(1). \tag{5.23}$$

The matrix stress resultants are then determined from equations (4.36) and (5.21)–(5.23) as

$$
\begin{aligned}
\boldsymbol{M}_{\beta\alpha} &= \frac{-2E}{1-\nu^2}\left\{\sum_{n=1}^{\infty} [(n-\nu)(\Delta^{n-1}u_3)_{,\beta\alpha}+\nu\delta_{\beta\alpha}\Delta^n u_3]\left(\frac{\pi}{2H}\mathfrak{D}\right)^{-2n}\mathbf{Z}(1)-\frac{H}{2}(1-\nu)\varepsilon_{\beta\nu}\boldsymbol{\Psi}_{,\nu\alpha}\right\},\\
\boldsymbol{M}_{3\alpha} &= \frac{-2E}{1-\nu^2}\left\{\sum_{n=1}^{\infty} n(\Delta^n u_3)_{,\alpha}\left(\frac{\pi}{2H}\mathfrak{D}\right)^{-2n-1}\mathbf{Z}(1)-\frac{H(1-\nu)}{4}\varepsilon_{\alpha\beta}\frac{\pi}{2H}\mathfrak{D}\boldsymbol{\Psi}_{,\beta}\right\},\\
l\boldsymbol{M}_{\beta} &= \frac{-2E}{1-\nu^2}\left\{\sum_{n=1}^{\infty} n(\Delta^n u_3)_{,\beta}\left(\frac{\pi}{2H}\mathfrak{D}\right)^{-2n}\mathbf{Z}(1)-\frac{H(1-\nu)}{4}\varepsilon_{\beta\alpha}\left(\frac{\pi}{2H}\mathfrak{D}\right)^{2}\boldsymbol{\Psi}_{,\alpha}\right\}=\left(\frac{\pi}{2H}\mathfrak{D}\right)^{-1}\boldsymbol{M}_{3\beta},\\
l\boldsymbol{M}_{3} &= \frac{-2E}{1-\nu^2}\left\{\sum_{n=1}^{\infty} n(\Delta^{n+1}u_3)\left(\frac{\pi}{2H}\mathfrak{D}\right)^{-2n-1}\mathbf{Z}(1)\right\}.
\end{aligned}
\tag{5.24}
$$

Also the shear stress resultant from equations $(2.1)_3$, (2.14), $(4.6)_4$, $(4.14)_3$ and $(4.31)_2$ may be expressed as follows

$$
N_{3\beta} = \frac{h}{2}\boldsymbol{\sigma}^T\boldsymbol{m}_\beta = \frac{h}{2}\boldsymbol{\Sigma}^T\boldsymbol{M}_\beta = \frac{2}{H}\mathbf{Z}^T(1)\left(\frac{\pi}{2H}\mathfrak{D}\right)^{-2} l\boldsymbol{M}_\beta. \tag{5.25}
$$

Substitution of equations (5.24) and (5.6) into (5.25) yields

$$
\begin{aligned}
N_{3\beta} = &-\left(\frac{2}{H}\right)\frac{2E}{1-\nu^2}\left\{\sum_{n=1}^{\infty}\frac{(-1)^n n}{[2(n+1)]!}2^{2n+1}(2^{2n+2}-1)\left(\frac{\pi}{2}\right)^{2n+2}B_{2n+2}(\Delta^n u_3)_{,\beta}\right.\\
&\left.-\frac{H(1-\nu)}{4}\varepsilon_{\beta\alpha}\mathbf{Z}^T(1)\boldsymbol{\Psi}_{,\alpha}\right\}.
\end{aligned}
\tag{5.26}
$$

Expressions for the three-dimensional displacement field may also be determined in terms of $\boldsymbol{\Psi}$ and u_3 by a similar method, but they will not be recorded.

6. AN EXAMPLE

As an illustration in the use of this theory, the torsion of a flat plate of length $2l$ parallel to the coordinate direction X_2, width $2b$ parallel to the coordinate direction X_1 and uniform thickness will now be considered. The origin of the coordinate system is selected at the center of the plate and the nondimensional coordinates are defined by

$$
x_1 = \frac{X_1}{l}, \qquad x_2 = \frac{X_2}{l}, \qquad x_3 = \frac{2X_3}{h}. \tag{6.1}
$$

For P a finite integer a solution of the canonical equations (4.8)–(4.11) is sought subject to the following boundary conditions for the stress resultant and canonical matrix stress resultants

$$
N_{31} = 0, \qquad \boldsymbol{M}_{\beta 1} = \mathbf{0}, \qquad \boldsymbol{M}_{31} = \mathbf{0} \quad \text{for } x_1 = \pm\frac{b}{l}. \tag{6.2}
$$

The above follows from the boundary conditions for twist about the x_2 axis, namely

$$
t^*_{\alpha 1} = 0, \qquad t^*_{31} = 0 \quad \text{for } x_1 = \pm\frac{b}{l}. \tag{6.3}
$$

Consideration is restricted, for simplicity, to the static case and since there is no dependence upon x_2 for the function $\boldsymbol{\Psi}$ equation (4.8) reduces to

$$\frac{d^2\boldsymbol{\Psi}}{dx_1^2}-\frac{1}{H^2}\mathfrak{D}_1\boldsymbol{\Psi} = \mathbf{0}, \qquad H = \frac{h}{2l}. \tag{6.4}$$

The solution of (6.4) is given by

$$\boldsymbol{\Psi} = \left(\sinh\frac{x_1}{H}\sqrt{\mathfrak{D}_1}\right)\boldsymbol{A}+\left(\cosh\frac{x_1}{H}\sqrt{\mathfrak{D}_1}\right)\boldsymbol{B}, \tag{6.5}$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are $P\times 1$ constants of integration matrices, $\sinh(x_1/H)\sqrt{\mathfrak{D}_1}$ and $\cosh(x_1/H)\sqrt{\mathfrak{D}_1}$ are defined in terms of $\exp[(x_1/H)\sqrt{\mathfrak{D}_1}]$ in the usual manner, and $\sqrt{\mathfrak{D}_1} = \mathrm{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots \sqrt{\lambda_P})$. Also the solution of equations (4.9)–(4.11) for the problem under consideration is

$$\begin{aligned} u_3 &= \alpha l^2 x_1 x_2 \\ \boldsymbol{\Delta}_3 &= \mathbf{0} \\ \mathbf{X} &= -2\alpha l^2 H x_1 x_2 \mathfrak{D}_1^{-1}\mathbf{Z}(1), \end{aligned} \tag{6.6}$$

where α is the angle of twist per unit length. In addition, from equations (4.12), (4.17) and (6.6) it follows that

$$\boldsymbol{\Phi} = \mathbf{0}. \tag{6.7}$$

The canonical stress matrix resultants are determined from (4.14) and (6.5)–(6.7). The only nonvanishing stress matrix resultants are $\boldsymbol{M}_{21}$, $\boldsymbol{M}_{32}$ and $\boldsymbol{M}_2$. In particular $\boldsymbol{M}_{21}$ is given by

$$\boldsymbol{M}_{21} = -\frac{EH}{1+\nu}\left\{2\alpha l^2 H\mathfrak{D}_1^{-1}\mathbf{Z}(1)+\frac{1}{2H^2}\mathfrak{D}_1\left(\sinh\frac{x_1}{H}\sqrt{\mathfrak{D}_1}\right)\boldsymbol{A}+\frac{1}{2H^2}\mathfrak{D}_1\left(\cosh\frac{x_1}{H}\sqrt{\mathfrak{D}_1}\right)\boldsymbol{B}\right\}. \tag{6.8}$$

The shear stress resultant is determined from equation (2.14), which is expressed in terms of the canonical representation as

$$N_{3\beta} = lH\boldsymbol{\Sigma}^T\boldsymbol{M}_\beta. \tag{6.9}$$

Since $\boldsymbol{M}_1 = \mathbf{0}$ it follows that $N_{31} = 0$ so that the boundary condition $(6.2)_1$ is satisfied. Also, $\boldsymbol{M}_{11} = \mathbf{0}$ and $\boldsymbol{M}_{31} = \mathbf{0}$ so that the only nontrivial boundary condition is $(6.2)_2$ with $\beta = 2$. From equation (6.8) and the nontrivial boundary condition of (6.2) the following constants of integration are obtained

$$\boldsymbol{A} = \mathbf{0}, \qquad \boldsymbol{B} = -4\alpha l^2 H^3\left(\cosh\frac{b}{lH}\sqrt{\mathfrak{D}_1}\right)^{-1}\mathfrak{D}_1^{-2}\mathbf{Z}(1), \tag{6.10}$$

and hence

$$\boldsymbol{\Psi} = -4\alpha l^2 H^3\left(\cosh\frac{x_1}{H}\sqrt{\mathfrak{D}_1}\right)\left(\cosh\frac{b}{lH}\sqrt{\mathfrak{D}_1}\right)^{-1}\mathfrak{D}_1^{-2}\mathbf{Z}(1). \tag{6.11}$$

Of interest is the approximate three-dimensional displacement field given by equation (4.15) and it is determined, with the aid of equations (6.6) and (6.11), as

$$u_1^* = -\alpha l^2 H x_2 x_3 = -\alpha X_2 X_3 \tag{6.12}$$

$$u_2^* = \alpha l^2 H\left\{-x_1x_3+4H\mathbf{Z}^T(x_3)\left(\sinh\frac{x_1}{H}\sqrt{\mathfrak{D}_1}\right)\left(\cosh\frac{b}{lH}\sqrt{\mathfrak{D}_1}\right)^{-1}\mathfrak{D}_1^{-\frac{3}{2}}\mathbf{Z}(1)\right\} \tag{6.13}$$

$$u_3^* = \alpha l^2 x_1 x_2 = \alpha X_1 X_2, \tag{6.14}$$

where the following identity has been used

$$\mathbf{Z}^T\mathfrak{D}_1^{-1}\mathbf{Z}(1) = \tfrac{1}{2}\mathbf{Z}^T\mathbf{\Sigma} = \tfrac{1}{2}x_3. \tag{6.15}$$

Thus for all values of $P \geq 2$ u_1^* and u_3^* are the same as that predicted by the theory of elasticity. Equation (6.13) written out is

$$u_2^* = \alpha l^2 H\left\{-x_1x_2+4H\sum_{N=1}^{P}\frac{Z_N(1)}{\lambda_N^{\frac{3}{2}}}Z_N(x_3)\frac{\sinh(x_1/H)\sqrt{\lambda_N}}{\cosh(b/lH)\sqrt{\lambda_N}}\right\}, \tag{6.16}$$

where $Z_N(x_3)$ and λ_N are determined in Section 4. For the case when P tends to infinity, this equation becomes

$$u_2^* = \alpha l^2 H\left\{-x_1x_3+4H\sum_{N=1}^{\infty}\frac{(-1)^{N-1}}{\omega_N^3}\sin\omega_N x_3\frac{\sinh(\omega_N x_1/H)}{\cosh(\omega_N b/lH)}\right\}, \tag{6.17}$$

where $\omega_N = (2N-1)\pi/2$ and is the same as that predicted by the theory of elasticity.

Consider now the case when $P = 2$. The second term in equation (6.16) is given by

$$4H\lambda_1^{-\frac{3}{2}}Z_1(x_3)Z_1(1)\sinh\frac{x_1}{H}\sqrt{\lambda_1}/\cosh\frac{b\sqrt{\lambda_1}}{lH}, \tag{6.18}$$

and

$$Z_1(x_3) = 1{\cdot}5397x_3 - 0{\cdot}5386x_3^3 \doteq \sin\frac{\pi}{2}x_3$$

$$\sqrt{\lambda_1} = 1{\cdot}5712 \doteq \frac{\pi}{2} = 1{\cdot}5708$$

$$Z_1(1) = 1{\cdot}0011.$$

Thus the second term in equation (6.16) for $P = 2$ is close to the second term in the exact solution. Also the third term is much smaller than the second term in this expression. Now as P increases the second term gets even more close to the second term of the exact solution (see Figs. 1, 3 and 4) and as P tends to infinity, they become equal. Similar remarks also apply for the third term, etc.

The above illustrative problem clearly shows the type of approximation being used as well as the role of the sequence of higher order approximations. Further, the role of the canonical functions and their increasing approximation to the eigenfunctions is also brought out in this example.

It is anticipated that in vibration problems the same type of results would also be true. For example, if one selects $P = 2$ and the plate is vibrating in the first thickness mode, the functions Z_1 and H_1, as well as λ_1 and μ_1, are very good approximations to the correct eigenfunctions and eigenvalues. Further, any shape up to order four in x_3 may be expressed in terms of the set $\{1, \mathbf{H}, \mathbf{Z}\}$. If one is interested in higher modes then P

should be taken at least one larger than that mode number. In addition, the thickness function expansion should be in terms of the eigenfunction when doing the three-dimensional problem and thus for an approximate theory it would be physically correct to use approximate eigenfunctions for the thickness expansion. Also the error between the equations of the theory of elasticity and equations (4.8)–(4.11) depends only on the integer P and as P tends to infinity they become equivalent to the equations of linear isotropic elasticity in terms of three-dimensional displacement potentials.

Finally, in the case of symmetric layered composite plates the exact mode shapes are not always known. The above method, with modifications in order to account for the layering, could give a method to obtain approximate mode shapes. The main change in the theory will take place in the form of the "material matrix coefficients", but the method of obtaining the canonical functions will be the same.

REFERENCES

[1] A. I. LURÉ, Theory of thick plates. *PMM*. **6**, 151–168 (1942).
[2] A. I. LURÉ, *Three-dimensional Problems of the Theory of Elasticity*. John Wiley (1964).
[3] O. K. AKSENTIAN and I. I. VOROVICH, The state of stress in a thin plate. *Appl. Math. Mech.* **27**, 1621 (1963).
[4] O. K. AKSENTIAN and I. I. VOROVICH, On the determination of stress concentrations on the basis of the applied theory. *Appl. Math. Mech.* **28**, 727 (1964).
[5] O. K. AKSENTIAN, On stress concentration in thick plates. *Appl. Math. Mech.* **30**, 1146 (1966).
[6] A. L. CAUCHY, Sur l'équilibre et le mouvement d'une plaque solide. *Exercices de Mathématique* **3**, 328 (1828).
[7] S. D. POISSON, Mémoire sur l'équilibre et le mouvement des corps élastique. *Mem. Acad. Sci.* **8**, 357 (1829).
[8] G. KIRCHHOFF, Über das Gleichgewicht und die Bewegung einer Elastischen Scheibe. *J. Angew. Math.* **40**, 51 (1850).
[9] J. N. GOODIER, On the problem of the beam and the plate in the theory of elasticity. *Trans. R. Soc. Canada* **32**, 65 (1938).
[10] A. E. GREEN, The elastic equilibrium of isotropic plates and cylinders. *Proc. R. Soc.* **195**, 533 (1949).
[11] J. B. ALBLAS, Theorie van de Driedimensionale Spanningstoestand in Een Doorboorde Plaal. H. J. Paris (1957).
[12] R. TIFFEN and P. G. LOWE, An exact theory of generally loaded elastic plates in terms of moments of the fundamental equations. *Proc. Lond. Math. Soc.* **13**, 653 (1963).
[13] R. D. MINDLIN and M. A. MEDICK, Extensional vibration of elastic plates. *J. appl. Mech.* **26**, 561 (1959).
[14] R. D. MINDLIN, High frequency vibrations of crystal plates. *Quart. appl. Math.* **19**, 51 (1961).
[15] P. M. NAGHDI, The theory of shells and plates, mechanics of solids II. *Handbuch der Physik* **VIa** 2. Springer (1972).
[16] A. E. GREEN, P. M. NAGHDI and W. L. WAINWRIGHT, A general theory of a Cosserat surface. *Arch. Ration. Mech. Anal.* **20**, 287 (1965).
[17] A. E. GREEN and P. M. NAGHDI, The linear theory of an elastic Cosserat plate. *Proc. Camb. Phil. Soc.* **63**, 537 (1967).
[18] A. E. GREEN, N. LAWS and P. M. NAGHDI, Rods, plates and shells. *Proc. Camb. Phil. Soc.* **64**, 895 (1968).
[19] A. E. GREEN and P. M. NAGHDI, Non-isothermal theory of rods, plates and shells. *Int. J. Solids Struct.* **6**, 209 (1970).
[20] J. W. PROVAN and R. C. KOELLER, On the theory of elastic plates. *Int. J. Solids Struct.* **6**, 933 (1970).
[21] L. MIRSKY, *An Introduction to Linear Algebra*. Oxford University Press (1961).
[22] E. D. RAINVILLE, *Infinite Series*. Macmillan (1967).
[23] T. KATO, *Perturbation Theory for Linear Operators*. Springer (1966).

(*Received* 10 *November* 1972; *revised* 7 *February* 1973)

Абстракт—Выводится теория изгмба линейных, упругих пластинок, в виде произвольной системы линейно независимых функций, образующей базу для параметра толщины. Получается каноническая система уравнений в виде функций матрицы напряжений, из которой можно определить все кинематические переменные, возникающие в результате напряжения и матрицы напряженний. Эта каноническая система уравнений пользуется приближенными конфигурациями поперечных собственных видов колебаний, которые получаются путем решения задачи на собственные значения для суммарных функций разложения.